ON THE CONSTRUCTION OF THE LIAPUNOV FUNCTIONS FROM THE INTEGRALS OF THE EQUATIONS FOR PERTURBED MOTION

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1. Let the equations for perturbed motion be

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, t) \qquad (i = 1, 2, \ldots, n) \tag{1.1}$$

It is known that these admit $p < n$ first integrals

$$U_1(x_1, \ldots, x_n, t), \ldots, U_p(x_1, \ldots, x_n, t)$$

which vanish for $x_1 = x_2 = \ldots = x_n = 0$.

If we succeed in finding a function $\phi(U_1, \ldots, U_p)$ of the known integrals, this function being definite with respect to the variables $x_1, \ldots, x_n$, then the stability of motion [1] can be deduced without bringing into consideration other properties of the equations for perturbed motion.

It is natural therefore to begin the investigation with an examination of the conditions under which the simplest function of the known integrals, this function being a function of fixed sign,

$$\psi(U_1, \ldots, U_p) = U_1^2(x_1, \ldots, x_n, t) + \ldots + U_p^2(x_1, \ldots, x_n, t)$$

is definite.

The following theorem holds.

Theorem 1. In order that there exists any definite function $\phi(U_1, \ldots, U_p)$, of the known integrals, it is necessary and sufficient that the function

$$\psi(U_1, \ldots, U_p) = U_1^2(x_1, \ldots x_n, t) + \ldots + U_p^2(x, \ldots, x_n, t)$$

be definite.

Necessity. Assume that a positive definite function $\phi(U_1, \ldots, U_p)$ exists, whereas the function $\psi(U_1, \ldots, U_p)$ is not definite.

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According to the assumption the function $\phi\ (U_1, \ldots, U_p) = f(x_1, \ldots, x_n, t)$ vanishes for $x_1 = x_2 = \ldots = x_n = 0$. Since all the functions $U_1, \ldots, U_p$ also vanish for the zero value of the x's, it follows that the function $\phi\ (U_1, \ldots, U_p)$ must necessarily vanish.

In fact, $U_1(0, \ldots, 0, t) = \ldots = U_p(0, \ldots, 0, t) = 0$; consequently, $\phi\ (0, \ldots, 0) = f(0, \ldots, 0, t) = 0$.

If the function $f(x_1, \ldots, x_n, t)$ is positive definite, there exists a positive definite function $f_1(x_1, \ldots, x_n)$, independent of the time, such that $f(x_1, \ldots, x_n, t) \geqslant f_1(x_1, \ldots, x_n)$ for all $t \geqslant t_0$.

If $\delta\ (\epsilon)$ is a positive minimum of the function $f_1(x_1, \ldots, x_n)$ on the small sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = \epsilon$, and if $\delta\ (\epsilon) > 0$, then there must exist a positive number $\delta_1(\delta)$ such that on the small sphere, $\psi\ (U_1, \ldots, U_p) \geqslant \delta_1[\delta(\epsilon)]$.

In fact, if such a number did not exist, the function $\psi\ (U_1, \ldots, U_p)$ could become as small as desired on the sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = \epsilon$ and, because of its continuous dependence on $U_1, \ldots, U_p$, the function $\phi\ (U_1, \ldots, U_p) = f(x_1, \ldots, x_n, t)$ could also become arbitrarily small. Since the last function, according to the assumption, is bounded from below on the sphere, the function $\psi\ (U_1, \ldots, U_p)$ satisfies the condition $\psi\ (U_1, \ldots, U_p) \geqslant \delta_1[\delta(\epsilon)] > 0$ and is positive definite.

The sufficiency follows from the statement of the problem.

2. Continuing our discussion of the problem, let us prove the following theorem.

Theorem. The function $\psi\ (U_1, \ldots, U_p)$ will be definite only when for at least one of the integrals, say, $U_i(x_1, \ldots, x_n, t)$, it is possible to find a pair of definite functions

$$r_i\,(x_1^2 + \ldots + x_n^2), \quad \rho_i\,(x_1^2 + \ldots + x_n^2)$$

such that

$$U_i^2\,(x_1, \ldots, x_n, t) > r_i$$

whenever

$$x_1^2 + \ldots + x_n^2 > 0$$

$$U_1^2 + \ldots + U_{i-1}^2 + U_{i+1}^2 + \ldots + U_p^2 < \rho_i\,(x_1^2 + \ldots + x_n^2)$$

Necessity. If $\psi\ (U_1, \ldots, U_p)$ is definite and $W(x_1, \ldots, x_n)$ is also definite and such that $\psi\ (U_1, \ldots, U_p) > W$ when $x_1^2 + \ldots + x_n^2 > 0$, then

$$\psi\,(U_1, \ldots, U_p) > \theta(x_1^2 + \ldots + x_n^2),$$

where $\theta\ (x_1^2 + \ldots + x_n^2)$ is the minimum of $W(x_1, \ldots, x_n)$ on the sphere $x_1^2 + \ldots + x_n^2 = \epsilon$, and also, as it is not difficult to show, is a continuous and definite function of the square of the radius. Since

$$\psi\,(U_1, \ldots, U_p) = U_1^2 + \ldots + U_p^2$$

we can put

$$r_i(x_1^2 + \ldots x_n^2) = \rho_i(x_1^2 + \ldots + x_n^2) = \frac{1}{2}\theta(x_1^2 + \ldots + x_n^2)$$

Sufficiency. The conditions of the theorem will be satisfied by the function

$$W = \min[\rho_i(x_1^2 + \ldots + x_n^2),\ r_i(x_1^2 + \ldots + x_n^2)]$$

which, obviously, will be continuous and definite, provided there exist continuous and definite functions

$$\rho_i(x_1^2 + \ldots + x_n^2), \qquad r_i(x_1^2 + \ldots + x_n^2)$$

From the proof it also follows that, if it is possible to select such a pair of functions for any one of the integrals, then it can also be selected for any other integral.

The practical significance of the mentioned theorem will become evident in that case for which $U_1, \ldots, U_p$ do not depend explicitly on time.

Corollary. If $U_1, \ldots, U_p$ do not depend explicitly on time, then, in order that $\psi(U_1, \ldots, U_p)$ be definite, it is necessary and sufficient that at least one of the functions $U_i(x_1, \ldots, x_n)$ assumes only positive values at all points, other than $x_1 = \ldots = x_n = 0$, for which

$$U_1(x_1, \ldots x_n) = \ldots = U_{i-1}(x_1, \ldots, x_n) =$$
$$= U_{i+1}(x_1, \ldots, x_n) = \ldots U_p(x_1, \ldots, x_n) = 0$$

Moreover, if the last condition is satisfied by at least one of the functions $U_i(x_1, \ldots, x_n)$, then it is satisfied by any other function.

The proof of this proposition is omitted.

The last result essentially simplifies the problem when, from any $p - 1$ equations

$$U_1 = \ldots = U_{i-1} = U_{i+1} = \ldots = U_p = 0$$

it is possible to express any $p - 1$ variables, say $x_{n-p+2}, \ldots, x_n$, in terms of $x_1, \ldots, x_{n-p+1}$:

$$x_{n-p+2} = f_1(x_1, \ldots, x_{n-p+1}) \ldots x_n = f_{p-1}(x_1, \ldots, x_{n-p+1})$$

If this can be done, then the problem of the definiteness of $\psi(U_1, \ldots, U_p)$ will be determined from the definiteness of the function

$$V(x_1, \ldots, x_{n-p+1}) = U_i(x_1, \ldots, x_{n-p+1}, f_1, \ldots, f_{p-1})$$

with respect to the variables $x_1, \ldots, x_{n-p+1}$. If, however, the above mentioned operation can be carried out, but with fewer variables, then the problem will be solved by examining the definiteness with respect to

$x_1, \ldots, x_{n-p+k}$ of the function

$$V_1(x_1, \ldots, x_{n-p+k}) = U_1^2(x_1, \ldots, x_{n-p+k}, f_1, \ldots, f_{p-k}) + \\ + U_k^2(x_1, \ldots, x_{n-p+k}, f_1, \ldots, f_{p-k})$$

which depends on less than n variables $x_1, \ldots, x_{n-p+k}$. Here

$$x_{n-p+k+1} = f_1(x_1, \ldots, x_{n-p+k}) \ldots x_n = f_{p-k}(x_1, \ldots, x_{n-p+k})$$

are the result of solving the equations $U_{k+1}(x_1, \ldots, x_n) = \ldots = U_p(x_1, \ldots, x_n) = 0$ with respect to the last $p - k$ variables.

3. Assume that the time-independent integrals $U_1, \ldots, U_p$, being holomorphic functions of the variables $x_1, \ldots, x_n$, are of the form

$$U_k = \alpha_1^k x_1 + \ldots + \alpha_n^k x_n + \sum_{i,j=1}^{n} \alpha_{ij}^k x_i x_j + X_k \qquad (k = 1, 2 \ldots p) \tag{3.1}$$

where α_i^k, α_{ij}^k are constants, and $X_1, \ldots, X_p$ functions which do not contain terms of lower degree than 3 with respect to the variables.

Consider the following cases: (a) the rank of the matrix (α_i^k) is p; (b) the rank of the matrix (α_i^k) is less than p. If the rank is p, then the linear forms

$$v_1 = \alpha_1^1 x_1 + \ldots + \alpha_n^1 x_n, \ldots, v_p = \alpha_1^p + \ldots + \alpha_n^p x_n$$

are independent of each other.

Taking them as the new variables instead of $x_1, \ldots, x_p$, rewrite (3.1) in the form

$$U_k = v_k + \sum_{i,j=1}^{p} \beta_{ij}^k v_i v_j + \sum_{\substack{i=1 \\ j=p+1}}^{\substack{p \\ n}} \beta_{ij}^k v_i x_j + \sum_{i,j=p+1}^{n} \beta_{ij}^k x_i x_j + X_k' \quad (k = 1, 2, \ldots, p) \tag{3.2}$$

where β_{ij}^k are constants, and $X_1', \ldots, X_p'$ are holomorphic functions of $v_1, \ldots, v_p, x_{p+1}, \ldots, x_n$ which do not contain terms of lower degree than 3 with respect to the variables.

If the first $p - 1$ equations of (3.2) are solved for $v_1, \ldots, v_{p-1}$ as power series of $U_1, \ldots, U_{p-1}, v_p, x_{p+1}, \ldots, x_n$, then this unique solution will assume the form

$$v_k = U_k - \sum_{i,j=1}^{p-1} \beta_{ij}^k U_i U_j - \sum_{\substack{i=1 \\ j=p+1}}^{\substack{p-1 \\ n}} \beta_{ij}^k U_i x_j - \sum_{j=p+1}^{n} \beta_{pj}^k v_p x_j - \sum_{ij=p+1}^{n} \beta_{ij}^k x_i x_j + Y_k$$
$$(k = 1, 2, \ldots, p-1)$$

where $Y_1, \ldots, Y_{p-1}$ are functions of the same type as $X_1, \ldots, X_p$.

If $U_1, \ldots, U_{p-1}$ are put equal to zero and the result so obtained is substituted into the last equation of (3.2), then

$$U_p^0 = v_p + \beta_{pp}^p v_p^2 + \sum_{j=p+1}^{n} \beta_{pj}^p v_p x_j + \sum_{i,j=p+1}^{n} \beta_{ij}^p x_i x_j + Z$$

is obtained, where Z is a function of v_p, x_{p+1}, ..., x_n, the degree being not less than 3.

From the last equation it is seen that U_p^0 can assume values of different signs, depending on the sign of v_p. Hence, on the basis of the Corollary of Section 2, we conclude that from the functions U_1, ..., U_p it is impossible to construct a definite integral.

4. If among the forms v_1, ..., v_p, $p - 1$ are independent, which corresponds to the case when the rank of the matrix (a_i^k) is $p - 1$, then these $p - 1$ linear forms can be taken for the new variables. Let such forms be v_1, ..., v_{p-1}, and let the form v_p in terms of them be of the form $v_p = \gamma_1 v_1 + \ldots + \gamma_{p-1} v_{p-1}$. Then the equations (3.2) assume the form

$$U_k = v_k + \sum_{i,j=1}^{p-1} \beta_{ij}^k v_i v_j + \sum_{\substack{i=1 \\ j=p}}^{\substack{p-1 \\ n}} \beta_{ij}^k v_i x_j + \sum_{i,j=p}^{n} \beta_{ij}^k x_i x_j + X_k' \quad (k = 1, 2, \ldots, p-1$$

$$U_p = \gamma_1 v_1 + \ldots + \gamma_{p-1} v_{p-1} + \sum_{i,j=1}^{p-1} \beta_{ij}^p v_i v_j + \sum_{\substack{i=1 \\ j=p}}^{\substack{p-1 \\ n}} \beta_{ij}^p v_i x_j + \sum_{i,j=p}^{n} \beta_{ij}^p x_i x_j + X_p'$$

Solving the first $p - 1$ equations with respect to v_1, ..., v_{p-1}, we obtain

$$v_k = U_k - \sum_{i,j=1}^{p-1} \beta_{ij}^k U_i U_j - \sum_{\substack{i=1 \\ j=p}}^{\substack{p-1 \\ n}} \beta_{ij}^k U_i x_j - \sum_{i,j=p}^{n} \beta_{ij}^k x_i x_j + Y_k$$

Putting U_1, ..., U_{p-1} equal to zero in these equations, we obtain

$$v_k^0 = -\sum_{i,j=p}^{n} \beta_{ij}^k x_i x_j + Y_k^0 \qquad (k = 1, 2, \ldots, p-1) \tag{4.1}$$

where Y_1^0, ..., Y_{p-1}^0 are functions of Y_1, ..., Y_{p-1} when U_1, ..., U_{p-1} are all equal to zero.

Substituting the expressions (4.1) into the last equation of (3.2) we obtain

$$U_p^0 = -\gamma_1 \sum_{i,j=p}^{n} \beta_{ij}^1 x_i x_j - \ldots - \gamma_{p-1} \sum_{i,j=p}^{n} \beta_{ij}^{p-1} x_i x_j + \sum_{i,j=p}^{n} \beta_{ij}^p x_i x_j + Z$$

where Z is a holomorphic function with respect to x_p, ..., x_n, of degree not lower than three.

If the quadratic form

$$R=\sum_{i,\,j=p}^{n}(-\gamma_1\beta_{ij}^{1}-\ldots-\gamma_{p-1}\beta_{ij}^{p-1}+\beta_{ij}^{p})x_ix_j$$

is definite with respect to x_p, ..., x_n, then the function $\psi(U_1, \ldots, U_p) = U_1^2 + \ldots + U_p^2$ will also be definite, and the unperturbed motion stable.

If, however, this form is of variable sign, then it is impossible to construct from the given integrals a definite function.

If the rank of the matrix (a_i^k) is $p - r$, then replace $x_1, \ldots, x_{p-r}$ in the equations (3.1) by the new variables $v_1, \ldots, v_{p-r}$ which are linearly independent linear forms. Solving the first $p - r$ equations with respect to $v_1, \ldots, v_{p-r}$, we obtain

$$v_k=U_k-\sum_{i,\,j=1}^{p-r}\beta_{ij}^{k}U_iU_j-\sum_{\substack{i=1\\ j=p-r+1}}^{p-r,\ n}\beta_{ij}^{k}U_ix_j-\sum_{i,\,j=p-r+1}^{n}\beta_{ij}^{k}x_ix_j+Y_k \qquad (k=1,\ldots,p-r)$$

Putting $U_1, \ldots, U_{p-r}$ equal to zero, we obtain

$$v_k^0=-\sum_{i,\,j=p-r+1}^{n}\beta_{ij}^{k}x_ix_j+Y_k^0 \qquad (k=1,\ldots,p-r)$$

Assuming that $v_{p-r+1}, \ldots, v_p$ can be expressed in terms of $v_1, \ldots, v_{p-r}$ in the form

$$v_{p-r+1}=\gamma_1^1v_1+\ldots+\gamma_{p-r}^{1}v_{p-r},\ \ldots,\ v_p=\gamma_1^rv_1+\ldots+\gamma_{p-r}^{r}v_{p-r}$$

after the substitution of $v_1^0, \ldots, v_{p-r}^0$ into the last r equations of the system (3.2), we obtain

$$U_{p-r+k}^0=\sum_{i,\,j=p-r+1}^{n}(-\gamma_1^k\beta_{ij}^{1}-\ldots-\gamma_{p-r}^{k}\beta_{ij}^{p-r}+\beta_{ij}^{p-r+k})x_ix_j+X_{p-r+k}^0 \qquad (k=1,2,\ldots,r)$$

where $X_{p-r+1}^0, \ldots, X_p^0$ are functions of $X_{p-r+1}, \ldots, X_p$ when $U_1, \ldots, U_{p-r}$ are all equal to zero.

As it was shown in Section 2, the function $\psi(U_1, \ldots, U_p)$ will be definite only when the function

$$(U_{p-r+1}^0)^2+\ldots+(U_p^0)^2 \qquad (4.2)$$

is definite with respect to $x_{p-r+1}, \ldots, x_n$.

The expansion of the function (4.2) in terms of the powers of the variables begins with the form of degree 4.

$$S_{p-r}=\sum_{k=1}^{r}\Big[\sum_{i,\,j=p-r+1}^{n}(-\gamma_1^k\beta_{ij}^{1}-\ldots-\gamma_{p-r}^{k}\beta_{ij}^{p-r}+\beta_{ij}^{p-r+k})x_ix_j\Big]^2$$

For the definiteness of the function (4.2) it is sufficient that S_{p-r} be definite with respect to $x_{p-r+1}, \ldots, x_n$.

When the rank $p - r$ $(r > 0)$ of the matrix (a_i^k) does not change with time and the modulus of at least one of the minors of order $p - r$ of the given matrix always exceeds a certain constant, then the method outlined can be carried over completely to the case where a_i^k, a_{ij}^k and the remaining coefficients of the expansion are continuous and bounded functions of time.

Also it is not difficult to show that $\psi\ (U_1, \ldots, U_p)$ will not be definite when the indicated rank for at least one instant of the time $t \geqslant t_0$ becomes equal to p.

5. Consider now the question of the connection between the outlined method and Chetaev's [2] method of selecting a definite linear bundle.

Let us indicate briefly the method of Chetaev. If the given time-independent integrals are holomorphic functions of the variables, then the constants $\lambda_1, \ldots, \lambda_p, C_1, \ldots, C_p$ are selected in such a way that the expansion of the function

$$\chi(U_1, \ldots, U_p) = \lambda_1 U_1 + \ldots + \lambda_p U_p + C_1 U_1^2 + \ldots + C_p U_p^2$$

begins with a definite quadratic form.

By the theorem of Section 1 such constants can be selected only when the function $\psi\ (U_1, \ldots, U_p)$ is definite.

If the rank of the matrix (a_i^k) is $p - 1$, the λ_i and C_i can always be selected in such a way that the conditions for the definiteness of the quadratic form, with which the expansion of the function $\chi\ (U_1, \ldots, U_p)$ begins, coincide with the conditions for the definiteness of the quadratic form R with respect to $x_p, \ldots, x_n$. In fact, in this case the selection of $\lambda_1, \ldots, \lambda_p$ is uniquely possible except for a common factor.

Put

$$\lambda_1 = -\gamma_1, \ldots, \lambda_{p-1} = -\gamma_{p-1}, \qquad \lambda_p = 1$$

Then

$$K = \lambda_1 U_1 + \ldots + \lambda_p U_p = R + \sum_{i,j=1}^{p-1} (-\gamma_1 \beta_{ij}^1 - \ldots - \gamma_{p-1} \beta_{ij}^{p-1} + \beta_{ij}^p) v_i v_j +$$

$$+ \sum_{\substack{i=1 \\ j=p}}^{\substack{p-1 \\ n}} (-\gamma_1 \beta_{ij}^1 - \ldots - \gamma_{p-1} \beta_{ij}^{p-1} + \beta_{ij}^p) v_i x_j + Z'$$

where Z' denotes the totality of terms of order not less than 3, or, using some abreviated notations,

$$K = \sum_{i,j=1}^{p-1} \theta_{ij} v_i v_j + \sum_{\substack{i=1 \\ j=p}}^{\substack{p-1 \\ n}} \theta_{ij} v_i x_j + \sum_{i,j=p}^{n} \theta_{ij} x_i x_j + Z'$$

If R is definite, then*

$$\theta_{nn} > 0, \qquad \begin{vmatrix} \theta_{nn} & \theta_{n,\,n-1} \\ \theta_{n-1} & \theta_{n-1,\,n-1} \end{vmatrix} > 0, \ldots, \begin{vmatrix} \theta_{nn} & \ldots & \theta_{np} \\ \ldots & \ldots & \ldots \\ \theta_{pn} & \ldots & \theta_{pp} \end{vmatrix} > 0$$

and, in addition, $\theta_{n-1,\,n-1} > 0, \ldots, \theta_{pp} > 0$.

Since

$$\chi(U_1, \ldots, U_p) = K + C_1' U_1^2 + \ldots + C_{p-1} U_{p-1}^2$$

(the summation is to be performed up to $p - 1$, since v_p depends linearly on $v_1, \ldots, v_{p-1}$), it is clear that $C_1, \ldots, C_{p-1}$ can always be selected sufficiently large so that the inequalities

$$\begin{vmatrix} \theta_{nn} & \ldots & \theta_{n,\,p-1} \\ \ldots & \ldots & \ldots \\ \theta_{p-1,\,n} & \ldots & \theta_{p-1,\,p-1} + C_{p-1}' \end{vmatrix} > 0 \ldots, \quad \begin{vmatrix} \theta_{nn} & \ldots & \theta_{n1} \\ \ldots & \ldots & \ldots \\ \theta_{1n} & \ldots & \theta_{11} + C_1' \end{vmatrix} > 0$$

are satisfied.

The fulfilment of these inequalities is sufficient in order for the expansion of the function $\chi(U_1, \ldots, U_p)$ to begin with a definite quadratic form.

A similar statement, however, cannot be made if the rank of the matrix (a_i^k) is less than $p - 1$.

It is easy to show that from the known integrals a definite bundle can be constructed only when it can be constructed from any system of p integrals, each of which is a linear combination of the initially given integrals, and the linear substitution is non-singular.

In what follows it is more convenient to consider not the initially given system of integrals but the following linear substitution of them, which is, obviously, non-singular:

$$V_1 = U_1, \ldots, V_{p-r} = U_{p-r}, \ V_{p-r+1} = -\gamma_1^1 U_1 - \ldots - \gamma_{p-r}^1 U_{p-r} + U_{p-r+1}$$

$$V_p = -\gamma_1^r U_1 - \ldots - \gamma_{p-r}^r U_{p-r} + U_p$$

Since the expansions of the first $p - r$ integrals begin with linearly independent forms, it is clear that $\lambda_1 = \ldots = \lambda_{p-r} = 0$.

Consequently, in order to construct a bundle, the expansion of which begins with a definite quadratic form, it is necessary and sufficient that the function

* The coefficients are assumed to be symmetric.

$$T_0 = \lambda_{p-r+1} V_{p-r+1}^0 + \ldots + \lambda_p V_p^0 =$$

$$= \lambda_{p-r+1} \sum_{i,\, j=p-r+1}^{n} (-\gamma_1^1 \beta_{ij}^1 - \ldots - \gamma_{p-r}^1 \beta_{ij}^{p-r} + \beta_{ij}^{p-r+1}) x_i x_j + \ldots$$

$$\ldots + \lambda_p \sum_{i,\, j=p-r+1}^{n} (-\gamma_1^r \beta_{ij}^1 - \ldots - \gamma_{p-r}^r \beta_{ij}^{p-r} + \beta_{ij}^p) x_i x_j + Z''$$

be definite with respect to $x_{p-r+1}, \ldots, x_n$.

Hence the problem is reduced to the possibility of selecting a definite linear combination of forms with which the expansions

$$V_{p-r+1}^0, \ldots, V_p^0$$

begin.

However, it is possible to give an example, where it is impossible to select such a linear combination, although the form S_{p-r} is definite.

As an example consider the pair of quadratic forms

$$M_1 = x_1^2 - x_2^2, \qquad M_2 = \sqrt{2}\, x_1 x_2$$

Any linear combination of these

$$M_1 + \lambda M_2 = x_1^2 + \lambda \sqrt{2}\, x_1 x_2 - x_2^2$$

is of variable sign. However, the form

$$S_2 = (x_1^2 - x_2^2)^2 + (\sqrt{2}\, x_1 x_2)^2 = x_1^4 + x_2^4$$

is, obviously, definite.

Before giving an example, let us formulate a rule for obtaining the form R for the case where

$$U_k = \alpha_1^k x_1 + \ldots + \alpha_n^k x_n + \sum_{i,j=1}^{n} \beta_{ij}^k x_i x_j + X_k \quad (k = 1, 2, \ldots, p)$$

are holomorphic and time-independent integrals, and the rank of the system of forms $v_k = a_1^k x_1 + \ldots + a_n^k x_n (k = 1, 2, \ldots, p)$ is equal to $p - 1$.

(a) Solving the system of equations $v_1 = \ldots = v_{p-1} = 0$ (under the assumption that the minor made up of the first $p - 1$ columns of the matrix of the system of forms $v_1, \ldots, v_{p-1}$ is different from zero), we obtain

$$x_1 = \delta_p^1 x_p + \ldots + \delta_n^1 x_n \ldots x_{p-1} = \delta_p^{p-1} x_p + \ldots + \delta_n^{p-1} x_n$$

(b) Solving the system of equations

$$\alpha_i^1\gamma_1 + \ldots + \alpha_i^{p-1}\gamma_{p-1} = \alpha_i^p \qquad (i = 1, 2, \ldots, p-1)$$

we find $\gamma_1, \ldots, \gamma_{p-1}$.

(c) Substituting into the forms

$$\sum_{ij=1}^{n} \alpha_{ij}^k x_i x_j \qquad (k = 1, 2, \ldots, p)$$

for $x_1, \ldots, x_p$ their expressions obtained after the operation (a), we obtain the forms

$$\sum_{ij=p}^{n} \beta_{ij}^k x_i x_j \qquad (k = 1, 2, \ldots, p).$$

Then R assumes the form

$$R = \sum_{i,j=p}^{n} (-\gamma_1 \beta_{ij}^1 - \ldots - \gamma_{p-1} \beta_{ij}^{p-1} + \beta_{ij}^p) x_i x_j$$

6. As an example let us prove by the method of N.G. Chetaev and by the proposed method a well-known theorem of Routh [3].

If the equations of motion of a mechanical system, which is conservative and holonomic, are written down in the Hamiltonian form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots n) \tag{6.1}$$

and the function H does not depend explicitly on time and the last $n - k$ generalized coordinates $q_{k+1}, \ldots, q_n$, then the system of equations (6.1) assumes $n - k + 1$ first integrals

$$p_{k+1} = \alpha_{k+1}, \ \ldots, \ p_n = \alpha_n, \qquad H = C$$

If for fixed $\alpha_{k+1}, \ldots, \alpha_n$ the system of constants

$$p_1^0, \ldots, p_k^0, \qquad q_1^0, \ldots, q_k^0$$

represents a solution of the system of finite equations

$$\frac{\partial H}{\partial p_1} = \ldots = \frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial q_1} = \ldots = \frac{\partial H}{\partial q_k} = 0$$

then the equations (6.1) assume a solution of the form

$$p_1 = p_1^0, \ldots, \quad p_k = p_k^0, \quad q_1 = q_1^0, \ldots, q_k = q_k^0, \quad p_{k+1} = \alpha_{k+1}, \ldots, \ p_n = \alpha_n$$

$$q_{k+1} = \left(\frac{\partial H}{\partial p_{k+1}}\right)^0 (t - t_0), \ldots, \ q_n = \left(\frac{\partial H}{\partial p_n}\right)^0 (t - t_0) \tag{6.2}$$

It was shown by Routh that the kinetic energy T of the system in terms of the variables $p_{k+1}, \ldots, p_n$ and $\dot{q}_1, \ldots, \dot{q}_k$ has the form

$$T = \frac{1}{2} \sum_{i,j=1}^{k} \alpha_{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} \sum_{i,j=k+1}^{n} \beta_{ij} p_i p_j$$

i.e. it will not contain terms of the form $\theta \dot{q}_i p_j$.

By virtue of the previous discussion it is easy to show that, if

$$\zeta_i = \dot{q}_i - (\dot{q}_i)^0, \ldots, \qquad \xi_i = q_i - q_i^0 \qquad (i = 1, 2, \ldots, k)$$
$$\eta_i = p_i - \alpha_i \qquad (i = k+1, \ldots, n)$$

and H is a holomorphic function of its variables in the neighborhood of (6.2), and V denotes the potential energy of the system, then the expansion of the function $H - (H)^0$ in terms of the powers of ζ_i, ξ_i, η_i will be of the form

$$H - (H)^0 = \sum_{i=k+1}^{n} (\dot{q}_i)^0 \eta_i + \frac{1}{2} \sum_{i,j=1}^{k} [(\alpha_{ij})^0 + \gamma_{ij}] \zeta_i \zeta_j +$$
$$+ \frac{1}{2} \sum_{i,j=k+1}^{n} (\beta_{ij})^0 \eta_i \eta_j + \delta \left[\frac{1}{2} \sum_{i,j=k+1}^{n} \beta_{ij} \alpha_i \alpha_j + V\right]$$

Here γ_{ij} are functions which vanish when all the $\xi_1, \ldots, \xi_k$ are zero, and $\delta[f(q_1, \ldots, q_k)]$ denotes the variation of the function $f(q_1, \ldots, q_k)$ in the transition from the position $(q_1^0, \ldots, q_k^0)$ to the position $(q_1^0 + \xi_1, \ldots, q_k^0 + \xi_k)$.

Routh showed that, if for the values of (6.2) the function

$$F + V = \frac{1}{2} \sum_{i,j=k+1}^{n} \beta_{ij}(q_1, \ldots, q_k) \alpha_i \alpha_j + V(q_1, \ldots, q_k)$$

has an isolated minimum, then the motion is stable under the condition that the constants $\alpha_{k+1}, \ldots, \alpha_n$ are not perturbed. As it was shown by Liapunov [4], the last condition is not essential. Let us prove these propositions. Since the functions $H - (H)^0$, $\eta_{k+1}, \ldots, \eta_n$ are integrals of the equations of a perturbed motion and the $(\dot{q}_i)^0$ are constants, the function

$$M = H - (H)^0 - \sum_{i=k+1}^{n} (\dot{q}_i)^0 \eta_i + \sum_{i=k+1} C_i \eta_i^2$$

representing an integral of the system of equations of a perturbed motion, will for sufficiently large C_i be a positive definite function of $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_k, \eta_{k+1}, \ldots, \eta_n$, provided that the function

$$[H - (H)^0]_{\eta_{k+1} = \ldots = \eta_n = 0} = \sum_{i,j=1}^{k} [(\alpha_{ij})^0 + \gamma_{ij}] \zeta_i \zeta_j + \delta\left[\frac{1}{2} \sum_{i,j=k+1}^{n} \beta_{ij} \alpha_i \alpha_j + V\right]$$

is positive definite with respect to $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_k$.

The first sum will always be a positive definite function with respect to $\zeta_1, \ldots, \zeta_k$ for sufficiently small $|\xi_1|, \ldots, |\xi_k|$, because

$\gamma_{ij}(\xi_1, \ldots, \xi_k)$ vanish for $\xi_1 = \ldots = \xi_k = 0$, and

$$\sum_{i,j=1}^{k} (\alpha_{ij})^0 \zeta_i \zeta_j$$

represents a positive definite quadratic form with respect to the variables $\zeta_1, \ldots, \zeta_k$.

Hence the function M will be positive definite, and the motion stable, if the function $F + V$ has an isolated minimum for the values $q_1^0, \ldots, q_k^0$.

The proof is completed by the method of Chetaev.

Examination of the function

$$\psi[H - (H)^0, \eta_{k+1}, \ldots, \eta_n] = [H - (H)^0]^2 + \sum_{i=k+1} \eta_i^2$$

also leads us to the conclusion that it will be a positive definite function of ξ_i, ζ_i, η_i provided that the function

$$H - (H)^0]_{\eta_{k+1}=\ldots=\eta_n=0}$$

is positive definite with respect to the variables $\xi_1, \ldots, \xi_k$, $\zeta_1, \ldots, \zeta_k$. The latter will be positive definite if the function $F + V$ assumes an isolated minimum at the position $[(\dot{q}_i)^0, q_i^0]$.

This proof, as is easily seen, illustrates the case of Section 4, since $p = n - k + 1$ and the rank of the matrix of the linear parts of the known integrals is equal to $n - k$.

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